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A note on immersions of domains of fractional powers of certain sectorial operators in Sobolev spaces

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ABSTRACT

In this note we give a proof of a result on immersions of domains of fractional powers of certain sectorial operators associated to strongly elliptic operators in Sobolev spaces; such immersions preserve information on fractional derivatives. We also briefly comment on the application of this result to a problem of optimal control of mosquito populations.

1. Introduction

In many situations involving the study of semilinear parabolic partial differential equations by using semigroup arguments, it is usual that their solutions assume values in the domains of some power of the associated sectorial operator. In such cases, to proceed with the analysis it is generally helpful to count with some immersion result allowing to conclude that the values of such a solution also lies in some well identified Sobolev space; moreover, for technical reasons it would be very convenient that such Sobolev spaces preserved some information on some fractional derivatives of the solutions.

To describe a result of this sort and to precisely explain the class of operators that we consider in this note, we must firstly recall certain notations and facts. For this, let N and m be positive integers and $\Omega \subset R^N$ be bounded domain with a C^{2m}-boundary. Next, by using the notations as in [1, Chapter I, Section 19, pp. 73–80], we consider a uniformly strongly elliptic linear operator $A(x, D) = \sum_{|\beta|\leq 2m} a_\beta(x)D^\beta$ of order $2m$ in Ω and a system of m linear boundary operator $B(x, D) = \{B_j(x, D)\}_{j=1}^m$ on $\partial\Omega$, where $B_j(x, D) = \sum_{|\beta|\leq m_j} b^j_\beta(x)D^\beta$. Here, $D = (D_1, \dots, D_n)$ with $D_k = \partial/\partial x_k$, and we assume that the coefficients of $A(x, D)$ are continuous in $\overline{\Omega}$ and that the coefficients of B_j belong to $C^{2m-m_j}(\partial\Omega)$. Moreover, we suppose that (A, B, Ω) is a regular elliptic boundary value problem, as defined in [1, p. 76], and satisfies the strong complementary condition, as defined in [1, p. 77]. Since $A(x, D)$ is strongly elliptic, it also satisfies condition (19.5) of Chapter I in [1, p. 77]. In this way, we meet all the requirements of Theorem 19.4 of Chapter I in [1, p. 78], and by denoting A_p the operator $A(x, D)$ acting on the functional space $X = L^p(\Omega)$, with domain $D(A_p) = \{v \in W^{2m}_p(\Omega); B(x, D)v = 0 \text{ in } \partial\Omega\} \subset X$, the previous arguments tell us that $D(A_p)$ is dense in X and that for some real k the operator $A_p + kI$, with domain $D(A_p)$, is a sectorial operator; see also [1, p. 101].

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We also assume that for any $f \in L^p(\Omega)$, $1 < p < +\infty$, there is a unique weak solution v of the elliptic problem

$$A(x, D)v(x) = f(x), \quad x \in \Omega,$$
$$B(x, D)v = 0, \quad x \in \partial\Omega.$$

Under the previous conditions, there also holds the L^p-regularity theory for this problem, that is, $v \in W_p^{2m}(\Omega)$ and $\|v\|_{W_p^{2m}} \le C\|f\|_{L^p(\Omega)}$ for some constant C independent of f. In this case, in the previous arguments we can take $k = 0$ and then conclude that A_p with the dense domain $D(A_p) = \{v \in W_p^{2m}(\Omega); B(x, D)v = 0 \text{ in } \partial\Omega\} \subset X$ is a sectorial operator in $X = L^p(\Omega)$, and this is the class of operators we consider in this note.

For this class of operators, given $0 < \alpha < 1$, we may consider the powers A_p^α (for details see [2, Theorem 1.4.3, p. 26]), and the Banach space $X_p^\alpha = (D(A_p^\alpha), \|.\|_{\alpha,p})$, with the graph norm $\|x\|_{\alpha,p} = \|x\|_X + \|A_p^\alpha x\|_X$.

By using these notations, we recall Theorem 1.6.1 in [2], in the special case of $p = q$, where we have the following immersion in Sobolev spaces: $X_p^\alpha \hookrightarrow W_p^k(\Omega)$, for an integer $k \ge 0$ such $k/2m < \alpha < 1$. However, since k is an integer, this loses a bit of the information on fractional derivatives.

The following result which is the main one in this note, in contrast preserves some information on higher fractional derivatives.

Theorem 1.1. *Let $m \ge 1$, Ω is a bounded open set of class C^{2m} in $\mathbb{R}^N$ and $p > 2m$ such that $\frac{p-1}{p} < \alpha < 1$. Then, $X_p^\alpha \hookrightarrow W_p^{2m(1-\frac{1}{p})}(\Omega)$.*

This improves a little the result in [2], at least for $p > 2m$ and $k/(2m) < (p-1)/p < \alpha < 1$, in the sense that it gives more information on fractional derivatives.

Before describing how to prove this theorem, let us comment that this kind of result is usually obtained with the help of suitable characterization of interpolation spaces. In fact, much is known on immersions and specific characterization of certain interpolations of domains of sectorial operators, as can be seen, for example, in [2–5]. By using this sort of results, one could prove the stated theorem for $m = 1$ as follows. Let $\theta \in]0, 1[$ and $q \in [1, \infty[$, and denote $(X, D(A_p))_{\theta,q}$ the real interpolation between X and $D(A_p)$ (see [6, ch. 1, 1.2]). It is known that $(L^p(\Omega), D(A_p))_{1-\frac{1}{p},p} \hookrightarrow W_p^{2-\frac{2}{p}}(\Omega)$ (see [5, p. 497]). On the other hand, the following relationship between X_p^α, for $0 < \alpha < 1$, and the interpolation spaces $(L^p(\Omega), D(A_p))_{\alpha,p}$ is known: $(L^p(\Omega), D(A_p))_{\alpha,1} \hookrightarrow X_p^\alpha \hookrightarrow (L^p(\Omega), D(A_p))_{\alpha,\infty}$ (for details, see e.g. the book by Lunardi [6, Proposition 2.2.15, p. 56].) Since by Triebel [7, Theorem 1.3.3] one knows that $(L^p(\Omega), D(A_p))_{\alpha,\infty} \hookrightarrow (L^p(\Omega), D(A_p))_{1-\frac{1}{p},p}$, we then conclude that the stated result holds for $m = 1$.

In this article, we will follow a very different approach to prove the theorem.

We will start by considering an auxiliary linear evolution equation associated to the given sectorial operator A_p, with initial data $u_o \in X_p^\alpha$. By using results on analytic semigroups, we know that this equation admits solutions; then, by using known regularity results for analytic semigroups, elliptic regularity and trace results, we will be able to show that in fact the initial data u_0 of the equation must in fact be in $W_p^{2m(1-\frac{1}{p})}(\Omega)$. The details of these arguments will be given in the next section. We briefly describe this application in the last section of this note.

We finally remark that Theorem 1.1 was used in [8,9] in the process of derivation of certain regularity estimates required to analyze a problem of optimal control of mosquito populations. In [8,9], Theorem 1.1 of the present note was used when $A = -\Delta + I$ and $B = \partial/\partial\eta$, with η denoting the external unitary normal to the boundary.

2. Proof of the main result

We will need the following proposition that can be found in [2, Theorem 3.5.2, p. 71]. Written with our previous notations it is stated as follows.

Proposition 2.1. *Suppose that X is a Banach space and consider a sectorial operator $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ such that its powers $\mathcal{A}^\alpha$ are well defined for $0 \le \alpha < 1$; consider also the Banach space $X^\alpha = (D(\mathcal{A}^\alpha), \|.\|_\alpha)$, with the graph norm $\|x\|_\alpha = \|x\|_X + \|\mathcal{A}^\alpha x\|_X$. Let $f : U \to X$ is locally Lipschitz in an open set $U \subset \mathbb{R} \times X_p^\alpha$, for some $0 \le \alpha < 1$. Suppose $u(.)$ is a solution on $(t_0, t_1]$ of $u_t + \mathcal{A}u = f(t, u)$, such that $u(t_0) = u_0$ and $(t_0, u_0) \in U$. Then, when $0 < \gamma < 1$, $t \to u_t(t) \in X^\gamma$ is locally Hölder continuous for $t_0 < t \le t_1$, with*

$$\|u_t(t)\|_\gamma \le C(t - t_0)^{\alpha-\gamma-1}$$

for some constant $C = C(\mathcal{A}, \gamma, \alpha, t_0, u_0)$ depending only on $\mathcal{A}$, γ, α, t_0, u_0.

Proof of Theorem 1.1. We want to show that $X_p^\alpha \subset W_p^{2m(1-\frac{1}{p})}(\Omega)$ and that there exist a constant $c > 0$ such that $\|v\|_{W_p^{2m(1-\frac{1}{p})}(\Omega)} \le c\|v\|_{\alpha,p}$, for all $v \in X_p^\alpha$.

For this, take an arbitrary $u_0 \in X_p^\alpha$. If $u_0 = 0$, the result is trivially true.

Next, suppose $u_0 \neq 0$ and consider the following auxiliary linear evolution problem, where A_p is as we described in the introduction:

$$u_t + A_p u = 0, \qquad u(0) = u_0. \tag{2.1}$$

Let $\{S(t)\}_{t\geq 0}$ be the family of analytic semigroups associated with $-A$ (see, [2, Example (4) and Theorem 1.3.4, p.20] and [10, Corollary 2.2]). It follows that (2.1) has a unique global solution, which is given by

$$u(t) = S(t)u_0.$$

By Proposition 2.1, if $0 < \gamma < 1$ and $t_0 = 0$, it follows that

$$\|u_t(t)\|_{\gamma,p} \leq C_2(\|u_0\|_{\alpha,p})t^{\alpha-\gamma-1}, \quad \text{for all } t > 0, \tag{2.2}$$

where $C_2 = C_2(\|u_0\|_{\alpha,p})$ is a positive constant.

Take γ, such that $0 < \gamma < \alpha + 1/p - 1$, this is possible since $\frac{p-1}{p} < \alpha < 1$. These two last inequalities imply that $p(\alpha - \gamma - 1) + 1 > 0$.

Integrating (2.2) over $(0, \widetilde{T}]$, we obtain

$$\int_0^{\widetilde{T}} \|u_t(s)\|^p_{L^p(\Omega)} ds \leq C_2(\|u_0\|_{\alpha,p}) \int_0^{\widetilde{T}} s^{(\alpha-\gamma-1)p} ds = \frac{C_2(\|u_0\|_{\alpha,p})}{p(\alpha-\gamma-1)+1} \widetilde{T}^{p(\alpha-\gamma-1)+1}. \tag{2.3}$$

As $\|u_0\|_{\alpha,p} > 0$, we can choose $\widetilde{T} = \widetilde{T}(u_0) > 0$ small enough, such that

$$0 < \widetilde{T} \leq 1 \quad \text{and} \quad \frac{C_2(\|u_0\|_{\alpha,p})\widetilde{T}^{p(\alpha-\gamma-1)+1}}{p(\alpha-\gamma-1)+1} \leq \|u_0\|^p_{\alpha,p}. \tag{2.4}$$

We stress that this is the key step in our proof because, although this previous choice (2.4) of $\widetilde{T}$ depends on u_0, this choice is done in such a way that the final constant c is in fact independent of u_0.

From (2.3) and (2.4) we have

$$\|u_t\|^p_{L^p(Q_{\widetilde{T}})} \leq \|u_0\|^p_{\alpha,p}. \tag{2.5}$$

From (2.1), we have that $u(t)$ satisfies the elliptic problem

$$A(x, D)u(t) = -u_t(t) \in L^p(\Omega),$$
$$B(x, D)u(t) = 0 \quad \text{on } \Omega,$$

for all $t \in (0, \widetilde{T}]$. It follows from the $L^p(\Omega)$-elliptic regularity that

$$\|u(t)\|^p_{W^{2m}_p(\Omega)} \leq C_3(\Omega)^p \|u_t(t)\|^p_{L^p(\Omega)}.$$

Next, by integrating this last inequality on $(0, \widetilde{T}]$ and using (2.5), we get

$$\|u\|_{L^p(0,\widetilde{T};W^{2m}_p(\Omega))} \leq C_3(\Omega)\|u_t\|_{L^p(0,\widetilde{T};L^p(\Omega))} \leq C_3(\Omega)\|u_0\|_{\alpha,p}. \tag{2.6}$$

By (2.5) and (2.6) we have $u \in W^{2m,1}_p(Q_{\widetilde{T}})$ and

$$\|u\|_{W^{2m,1}_p(Q_{\widetilde{T}})} \leq (1 + C_3(\Omega))\|u_0\|_{\alpha,p}. \tag{2.7}$$

From the trace theorem in [11, Theorem 1.1 for $p = q$] or [12, Chapter II, Lemma 3.4], we have that $u|_{t=0} = u_0 \in W^{2m(1-\frac{1}{p})}_p(\Omega)$, and also that

$$\|u_0\|_{W^{2m(1-\frac{1}{p})}_p(\Omega)} \leq C_4\|u\|_{W^{2m,1}_p(Q_{\widetilde{T}})}, \tag{2.8}$$

where C_4 is independent of t and u_0, for all $t \in [0, \widetilde{T}]$. In fact, as it can be seen in [11, Theorem 1.1], such a constant is of form $c^*\widetilde{T}^{1-1/p}$, where c^* is independent of t; by choosing $\widetilde{T}$ satisfying (2.4) we find the constant C_4 desired.

From (2.7) and (2.8) we conclude the result, i.e.

$$\|u_0\|_{W^{2m(1-\frac{1}{p})}_p(\Omega)} \leq c\|u_0\|_{\alpha,p},$$

where $c = C_4(1 + C_3(\Omega))$ is independent of u_0. As u_0 was taken arbitrary in X^α_p, the theorem is proved. □

3. Remarks on an application

We briefly describe an optimal control problem related to the issue of controlling mosquito populations in a bounded region $\Omega \subset \mathbb{R}^2$; this problem is studied in detail in [8] and [9]; here we just describe it to call the attention to the importance of the results obtained in the present article for the analysis performed in [8,9].

The objective is to find an optimal trajectory to be followed by a insecticide spraying unit in order to minimize a certain functional involving both the whole mosquito population and the spraying operational costs. All the admissible trajectories for the spraying unit start at a fixed location, which is normalized to be the origin; and is mathematically described as $\gamma : [0, T] \to \mathbb{R}^2$, where $0 < T < \infty$ is a fixed final time. An optimal trajectory γ^* is such that

$$F(\gamma^*) = \min\{F(\gamma) : \gamma \in \mathcal{A}\}, \tag{3.1}$$

where the set of admissible trajectories $\mathcal{A}$ and the functional $F(\gamma)$ are given by

$$\mathcal{A} = \{\gamma \in (H^1(0, T)) \times (H^1(0, T)) : \gamma(0) = (0, 0)\}, \tag{3.2}$$

$$F(\gamma) = J(\gamma, u) = \mu_0 \int_0^T |\gamma(t)|^2 dt + \mu_1 \int_0^T |\gamma'(t)|^2 dt + \mu_2 \int_Q |u(x, t)|^2 dxdt, \tag{3.3}$$

where $Q = \Omega \times (0, T)$; $\mu_0 \geq 0$, $\mu_1 > 0$ and $\mu_2 > 0$ are fixed constants, and $u = u(x, t)$ is the density of mosquitoes at position $x \in \Omega$ and time $t \in [0, T]$; u depends on the trajectory γ of the spraying unit through the following equations:

$$\begin{cases} \partial_t u - \nu \Delta u = g(x, t, u) - bk(x - \gamma(t))u & \text{in } Q = \Omega \times (0, T), \\ (\partial/\partial\eta)(u) = 0 & \text{in } S = \partial\Omega \times (0, T), \\ u(0, .) = u_0 & \text{in } \Omega, \end{cases} \tag{3.4}$$

where u_0 is a given initial mosquito population.

In (3.3), the first two integrals are related to the operational costs of the insecticide spraying; the last integral is related to the amount of mosquito population. In this model, the following are assumed: **(a)** the mosquito population grows at rate $g(x, t, u)$ and spreads with a constant diffusion coefficient $\nu > 0$. Verhustian growth, i.e., $g(x, t, u) = au(1 - u/M)$, with positive constants a and M, or more general growth rates are allowed; **(b)** the insecticide kills the mosquitoes at an effective rate which depends on the distance of the point of spraying application. More specifically, the killing rate at a point x when the spraying unit is at position $\gamma(t)$ is given by $bk(x-\gamma(t))$, where $0 \leq k(.) \leq 1$ is a given C^1-function with compact support, and the constant $b > 0$ is the maximum killing rate; **(c)** for simplicity, homogeneous Neumann boundary condition is considered; **(d)** also for simplicity, it is assumed that there are no obstacles to the admissible trajectories.

These assumptions can be relaxed (see [8,9]) where the existence of an optimal trajectory for (3.1)–(3.4) is proved and the associated first order optimality conditions characterizing such an optimal trajectory are also derived, which is important since they usually serve as guides for devising both feedback controls and efficient algorithms for numerical simulations. Being $\gamma^*(\cdot)$ an optimal trajectory and being $u^*(\cdot, \cdot)$ and $p^*(\cdot, \cdot)$, respectively, the associate optimal state (mosquito population) and the optimal adjoint state, and denoting $g_u(x, t, u^*)p^*$ the Gateaux-derivative of g at u^* in the direction of p^*, the following equations must be satisfied:

$$\begin{cases} \partial_t u^* - \alpha \Delta u^* = g(x, t, u^*) - b\, k(x - \gamma^*)u^* & \text{in } Q, \\ (\partial/\partial n)(u^*) = 0 & \text{on } S, \\ u^*(0) = u_0 & \text{in } \Omega, \\ -\partial_t p^* - \alpha \Delta p^* = g_u(x, t, u^*)\, p^* - b\, k(x - \gamma^*)p^* - u^* & \text{in } Q, \\ (\partial/\partial n)(p^*) = 0 & \text{on } S, \\ p^*(T) = 0 & \text{in } \Omega, \\ -\mu_1 \gamma^{*''} + \mu_0 \gamma^* - \mu_2 b \displaystyle\int_\Omega p^* u^* \nabla k(x - \gamma^*) dx = 0 & \text{in } (0, T), \\ \gamma^*(0) = 0, \quad \gamma^{*'}(T) = 0. \end{cases}$$

The proof of the existence of an optimal trajectory as done in [8,9] follows the usual procedure of considering a minimizing sequence. However, to complete the arguments, several technical questions have to be previously answered, specially the ones related to the dependence of the mosquito populations on given trajectories for the spraying unit, and suitable estimates are required to pass to the limit in the strong nonlinearities of the problem. In these arguments, the results of the present article are fundamental. The first order optimality conditions were obtained in [8,9] by using the Dubovitskii and Milyutin methodology; see for instance [13]. For this, the differentiability of the mosquito population with respect to trajectories was required, and the immersion results of the present article play a role.

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References

[1] A. Friedman, Partial Differential Equations, Robert E. Krieger Publishing Co., 1976.
[2] D. Henry, Geometric Theory of Semilinear Parabolic Equations, in: Lecture notes in Mathematics, vol. 840, Springer-Verlag, 1981.
[3] H. Hoshino, Y. Yamada, Solvability and smoothing effect for semilinear parabolic Equations, Funkt. Ekv. 34 (1991) 475–494.
[4] P. Cannarsa, V. Vespri, On maximal L^p regularity for the abstract Cauchy problem, Boll. Un. Mat. Ital. B 5 (1) (1986) 165–175.
[5] W. von Wahl, The equation $u' + A(t)u = f$ in a Hilbert space and L^p-estimates for parabolic equations, J. London Math. Soc. 25 (1982) 483–497.
[6] A. Lunardi, Analytic Semigroups and Optimal Regularity in Parabolic Problems, Birkhäuser, Basel, 1995.
[7] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, North-Holland, Amsterdam, 1978.
[8] A.L.A. de Araujo, Análise matemática de um modelo de controle de populações de mosquitos, Tese de Doutorado em Matemática, Universidade Estadual de Campinas, SP, Brazil 2010.
[9] A.L.A. de Araujo, J.L. Boldrini, A mathematical analysis of an optimal control problem of mosquito populations (2012) (in preparation).
[10] A. Pazy, Semigroups of Linear Operator and Applications to Partial Differential Equations, Springer-Verlag, 1983.
[11] P. Weidemaier, On the sharp initial trace of functions with derivates in $L_q(0, T; L_p(\Omega))$, Bolletino U. M. I. 7 (9 - B) (1995) 321–338.
[12] O. Ladyzhenskaya, V. Solonnikov, N. Uraltseva, Linear and Quasilinear Equations of Parabolic Type, Amer. Math. Soc., 1968.
[13] I.V. Girsanov, Lectures on Mathematical Theory of Extremum Problem, in: Lectures notes in Economics and Mathematical Systems, vol. 67, Springer-Verlag, Berlin, 1972.